Exam. Code : 211004
Subject Code : 4911

M.Sc. (Mathematics) 4th Semester (Batch 2020-22)

MATH-581 : FUNCTIONAL ANALYSIS—II

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt *five* questions in all, selecting at least *one* question from each section. The fifth question may be attempted from any section. All questions carry equal marks.

SECTION—A

1. (a) Define weak convergence. Give two examples. Prove that the notions of week and strong convergence are equivalent in finite dimensional normed linear spaces.

   (b) Prove that a bounded linear operator T on a Hilbert space H is self-adjoint if and only if $\langle x, Tx \rangle$ is real valued for all x in H.

2. Define adjoint of an operator, self adjoint operator, normal operator and unitary operator. Give one example in each case. Let T_1 and T_2 be normal operators. If T_1 commute with T_2^* and T_2 commute with T_1^*, then prove that T_1+T_2 and T_1T_2 are normal.

SECTION—B

3. (a) Define eigen values of a linear operator. Let $T : H \rightarrow H$ be a self adjoint operator. Prove that eigen values of T are real.

   (b) Let T be a compact operator on a Hilbert space $H \neq \{0\}$. Then prove that eigen values of T and the corresponding eigen space is finite dimensional.

4. (a) State and prove spectral theorem for normal operators.

   (b) Define spectrum of a bounded linear operator. If A is a non-singular operator, then prove that $\sigma(ATA^{-1}) = \sigma(T)$.

SECTION—C

5. (a) Let X be a normed linear space and $\{T_n\}$ be a sequence of compact linear operators on X. If T in G bounded linear operator on X such that $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$, then prove that T is compact.

   (b) If T is a normal operator on a Hilbert space H and x_1 and x_2 an eigen vector of T corresponding to distinct eigenvalues, then prove that $x_1 \perp x_2$.

6. Let $T : X \rightarrow Y$ be a bounded linear operator and $S : Y \rightarrow Z$ be another bounded linear operator. If one of them is compact, then prove that SOT is compact.

SECTION—D

7. (a) Define singular and regular elements in a Banach algebra. Let $\mathcal{A}$ be a Banach algebra then prove that for every $x \in \mathcal{A}$ with $\|x-e\| < 1$, is regular and $x^{-1} = e + \sum_{n=1}^{\infty}(e-x)^n$

   (b) Let $\mathcal{A}$ be a Banach algebra with identity and $x \in \mathcal{A}$. Then prove that the spectral radius :

$$r(x) = \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}}$$

8. (a) Let $\mathcal{A}$ be a Banach algebra. Then prove that the set of singular elements in $\mathcal{A}$ is a subset of all topological divisors of zero.

   (b) If $\mathcal{A}$ is a Banach algebra with identity then prove that for $0 \neq x \in X$, the spectrum $\sigma(x) \neq \phi$. If $\mathcal{A} = M_r(\mathbb{R})$ and $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then find $\sigma(A)$.